WHO again?

Einstein's Equations in Electromagnetic Media

Encoding General Relativity Inside Optical Materials

Eren Erberk Erkul

Monday, 7 July 2025

Caltech TAPIR

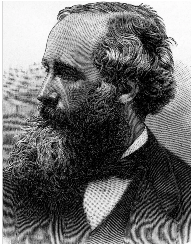
Invisibility Devices

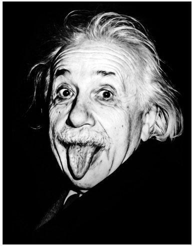
$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}, \quad \nabla \cdot \boldsymbol{B} = 0, \quad \nabla \times \boldsymbol{H} = \partial_t \boldsymbol{D} + \boldsymbol{J}, \quad \nabla \cdot \boldsymbol{D} = \rho$$

The covariant free-space Maxwell equations are equivalent to electromagnetism in a material medium (Tamm, 1924; Plebanski, 1960).

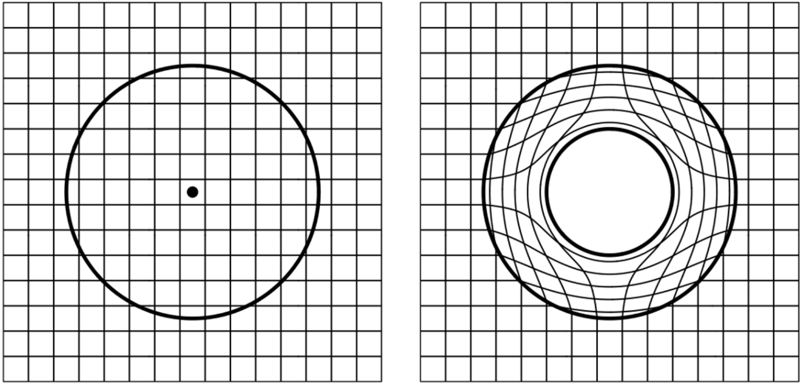
$$\boldsymbol{D} = \varepsilon\, \boldsymbol{E} + \frac{\boldsymbol{w}}{c} \times \boldsymbol{H}, \qquad \boldsymbol{B} = \frac{\mu}{\varepsilon c^2}\, \boldsymbol{H} - \frac{\boldsymbol{w}}{c} \times \boldsymbol{E}$$

$$\varepsilon^{ij} = \mu^{ij} = \frac{\sqrt{-g}}{g_{00}}\, g^{ij}, \qquad w_i = \frac{g_{0i}}{g_{00}}$$





Virtual vs. Physical Space

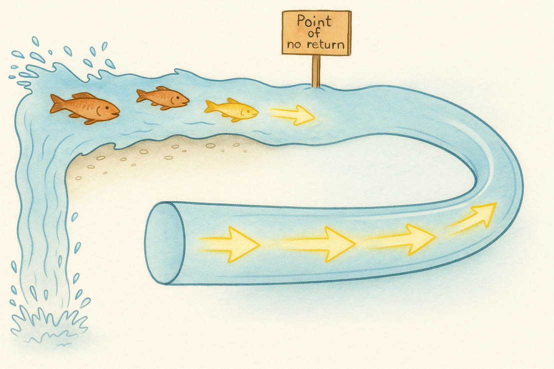


A coordinate deformation (left) is realised by a graded $\varepsilon_{ij}, \mu_{ij}$ profile (right). Light rays follow geodesics of the *virtual* metric while propagating through an ordinary laboratory sample.

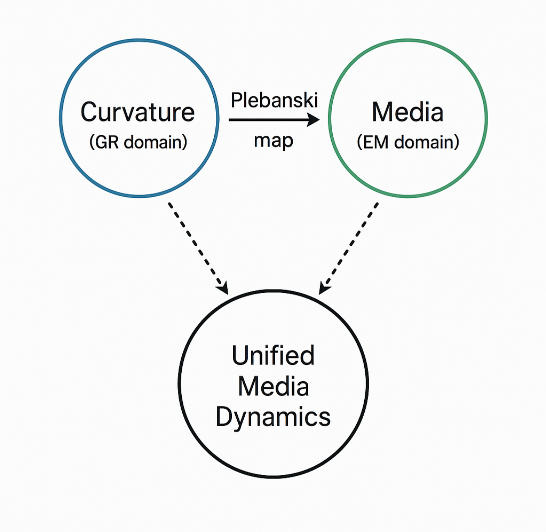
Spacetime transformations in moving media

$$w_i = \frac{g_{0i}}{g_{00}} \quad \Longrightarrow \quad w_i^{\text{lab}} = \frac{n^2 - 1}{1 - u^2 n^2 / c^2}\, u_i$$

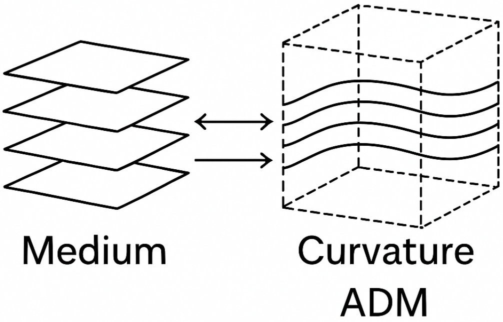
Motion couples **E** and **B** through g_{0i}. Flow, rotation, or sound waves mimic frame dragging.



The Big Picture



The Idea



Optical Metric in ADM Form

Transformation optics

$$ds^2 = g_{00}\, dt^2 + 2g_{0i}\, dt\, dx^i + g_{ij}\, dx^i dx^j$$

ADM $(3+1)$ split

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

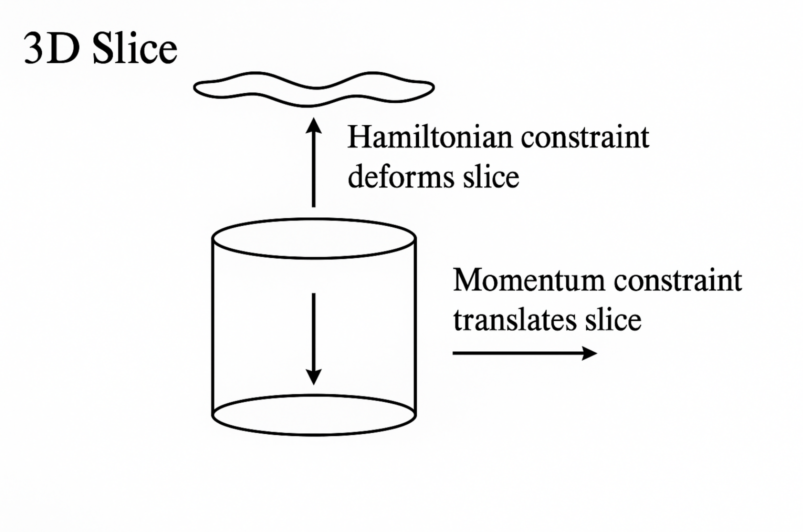
Identify the optical metric parameters via $g_{00} = -\alpha^2 + \gamma_{ij}\beta^i\beta^j,\ g_{0i} = \gamma_{ij}\beta^j,\ g_{ij} = \gamma_{ij}$.

ADM Constraint Equations

$$ {}^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi G\,\rho, \qquad D_j\big(K^{ij} - \gamma^{ij}K\big) = 8\pi G\, j^{\,i} $$

ρ and j^i are supplied by the electromagnetic energy density and Poynting vector once the mapping is complete.

Visualising the ADM Constraints



ADM Evolution Equations

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i,$$

$$\partial_t K_{ij} = -D_i D_j \alpha + \alpha\big({}^{(3)}R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j\big) + \dots$$

Set $g_{00} = -1$ ($\alpha = 1$) and identify the optical metric

$$\gamma_{ij}(\mathbf{x}) = \left(\det \varepsilon\, \varepsilon^{ij}\right)^{-1}.$$

Self-Gravity of Light

Electromagnetic field as source:

$$\rho = \frac{1}{2c^2}(\boldsymbol{D}\cdot\boldsymbol{E} + \boldsymbol{B}\cdot\boldsymbol{H}), \qquad \boldsymbol{j} = \frac{1}{c^2}\boldsymbol{E}\times\boldsymbol{H}.$$

This scenario naturally arises, e.g., in the early radiation-dominated universe.

Weak-Field Approximation

Assume

$$g_{ij} = \delta_{ij} + h_{ij}, \qquad |h_{ij}| \ll 1.$$

Gauge choice $\beta^i = 0$ gives

$$K_{ij} = -\tfrac{1}{2}\,\partial_t h_{ij}.$$

Linearising the constraints then leads to familiar wave equations.

Optical Analogue of Gravitational Waves

Hamiltonian constraint

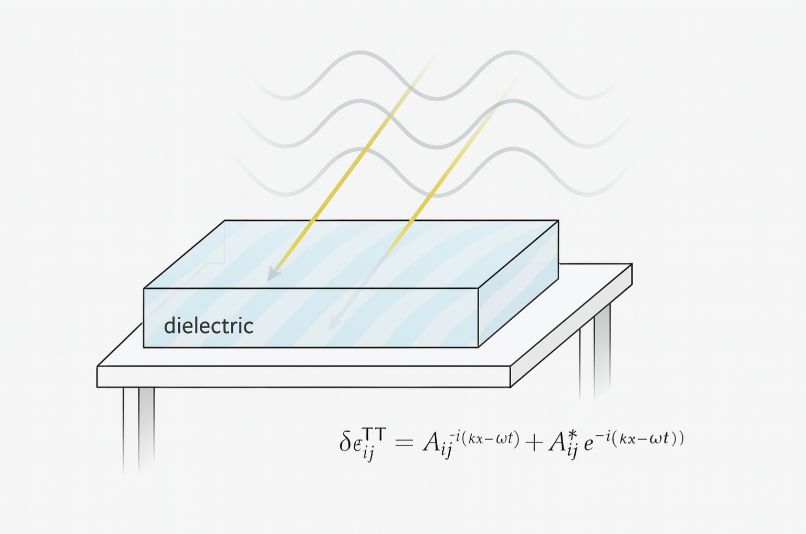
$$\partial_j \partial_k h^{jk} - \nabla^2 h = 0, \qquad \text{Momentum}: \ \partial_j \dot{h}^{ij} - \partial^i \dot{h} = 0.$$

TT-gauge isolates two radiative degrees:

$$\frac{1}{c^2} \partial_t^2 h_{ij}^{\mathrm{TT}} - \nabla^2 h_{ij}^{\mathrm{TT}} = 0.$$

Identify $h_{ij}^{\mathrm{TT}} = \delta \varepsilon_{ij}^{\mathrm{TT}}$ to obtain the optical wave equation for dielectric perturbations.

Plane-Wave Solution



Another Einstein$\leftrightarrow$Maxwell Lens: DGREM (Most et al. 2022)

Maxwell	$\Longleftrightarrow$	**DGREM**
$A_\mu \xrightarrow{d} F_{\mu\nu} = dA$		$\theta^{\hat{a}} \xrightarrow{d} F^{\hat{a}} = d\theta^{\hat{a}}$
$d \star F = \star J$		$d\, u^{\hat{a}} = t^{\hat{a}} + \kappa T^{\hat{a}}$
homogeneous / inhomogeneous		Nester–Witten / Sparling

after Phys. Rev. D 105 (2022) 124038

Why DGREM helps simulators

First-order & hyperbolic

$$\partial_t D_k^{\hat{a}} \;-\; \nabla \times H^{\hat{a}} \;=\; -\,J_k^{\hat{a}}$$

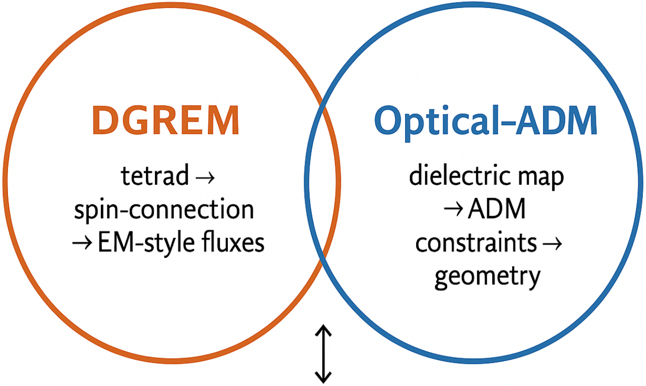
Flux-conservative core

$$\partial_t U \;+\; \nabla \cdot F(U) \;=\; 0$$

Machine-precision constraints

$$\nabla \cdot D^{\hat{a}} = \rho^{\hat{a}}, \qquad \nabla \cdot B^{\hat{a}} = 0$$

Two roads, one goal



Both recast GR in electromagnetic language—
—from opposite directions.

Take-Home Message

Gravity $\longleftrightarrow$ **Optics**

Transformation optics + ADM split

$$g_{\mu\nu} \quad \longleftrightarrow \quad \left(\varepsilon_{ij},\, \mu_{ij},\, \boldsymbol{w}\right)$$

Weak-field limit $\Rightarrow$ tabletop gravitational waves.

Design materials $\equiv$ *design geometry.*

Acknowledgement

This work was carried out under the mentorship and companionship of

Prof. Ulf Leonhardt

Acknowledgement

I gratefully thank

Prof. Saul Teukolsky & Prof. Elias Most

for giving me the opportunity to be a visiting researcher at TAPIR.