Einstein's Equations in Electromagnetic Media

Encoding General Relativity Inside Optical Materials

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Entropy

$$S = -\sum_i p_i \log p_i$$

Telegrapher's Equation

$$LC\,\partial_t^2 V + (RC + LG)\,\partial_t V + RG\,V - \partial_x^2 V = 0$$

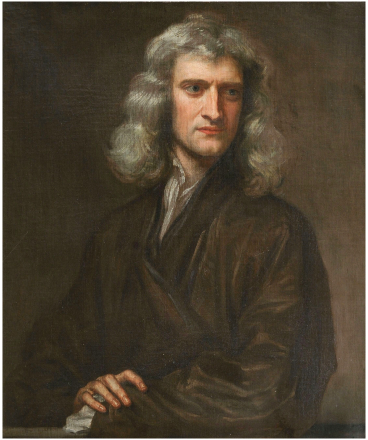
$$\tau\,\partial_t^2 \Phi + \partial_t \Phi - D\,\nabla^2 \Phi = 0$$

Holography Inspired Object Detection

Requiem for Gravity

From Newton to Einstein

1687: The First Theory of Gravity



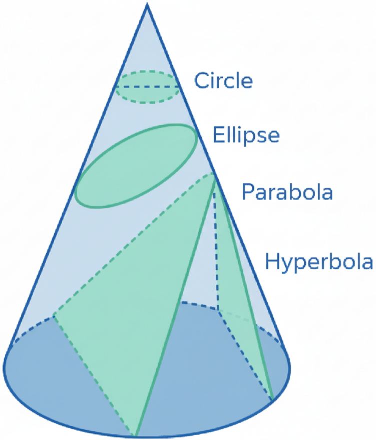
1687: The First Theory of Gravity

$$F = -\frac{GMm}{r^2}$$

The Orbits of the Planets

$$F = -\frac{GMm}{r^2}$$

$\Rightarrow$ conic-section trajectories



Orbits correspond to *conic sections*.

Speed of Light

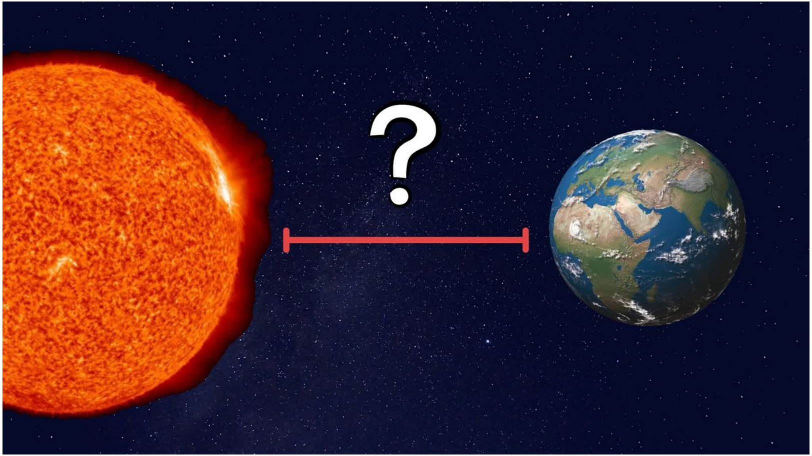
$$c = 299\,792\,458 \ \mathrm{m\,s^{-1}}$$

The Addition of Velocities

$$v_{\text{new}} = \frac{v + u}{1 + \frac{uv}{c^2}}$$

$v =$ speed of object · $u =$ your speed

Einstein's Question



How fast does gravity move?

Einstein's Field Equations

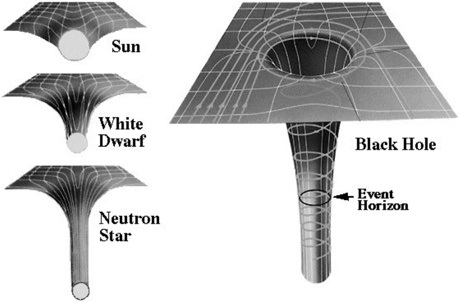
$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

The Mathematics of Curved Space

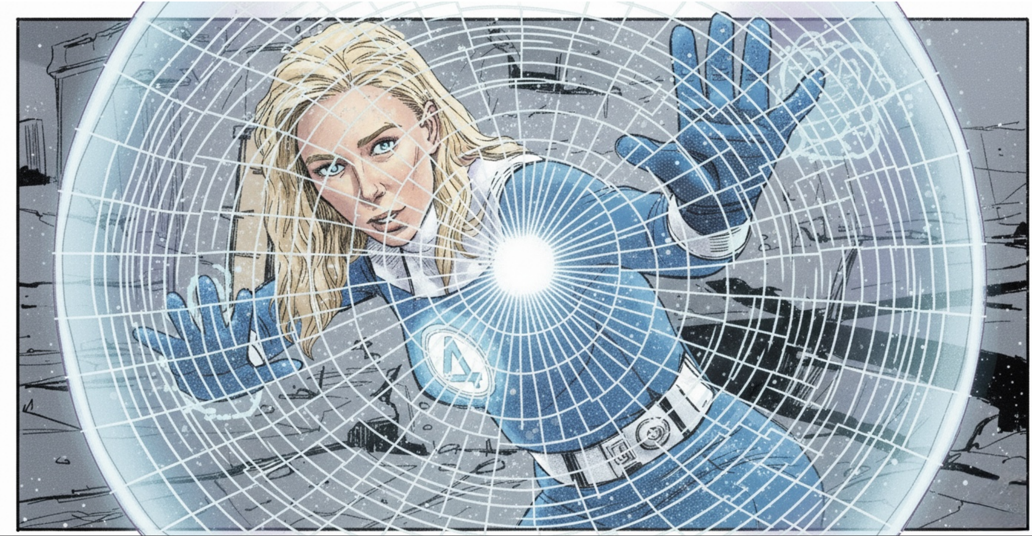
$$ds^2 = -f(r)\,c^2 dt^2 + f(r)^{-1} dr^2 + r^2 d\Omega^2$$

$$f(r) = 1 - \frac{2GM}{rc^2}, \quad d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2.$$

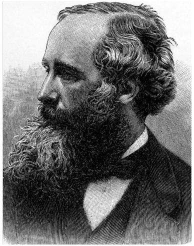
Schwarzschild geometry.

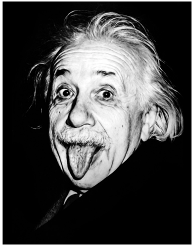


Invisibility Devices



$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}, \quad \nabla \cdot \boldsymbol{B} = 0, \quad \nabla \times \boldsymbol{H} = \partial_t \boldsymbol{D} + \boldsymbol{J}, \quad \nabla \cdot \boldsymbol{D} = \rho$$





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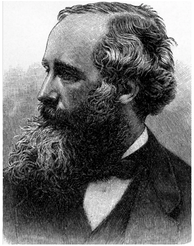
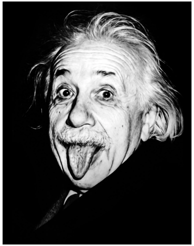
$$\boldsymbol{D} = \varepsilon_0 \, \boldsymbol{\varepsilon} \, \boldsymbol{E} + \frac{\boldsymbol{g}}{c} \times \boldsymbol{H}, \qquad \boldsymbol{B} = \mu_0 \, \boldsymbol{\mu} \, \boldsymbol{H} - \frac{\boldsymbol{g}}{c} \times \boldsymbol{E}$$





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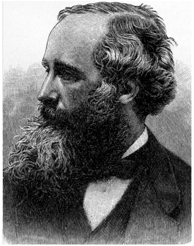
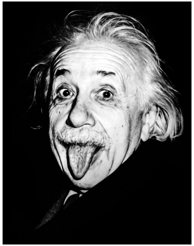
$$\varepsilon^{ij} = \mu^{ij} = -\frac{\sqrt{-g}}{g_{00}}\, g^{ij}, \qquad g_i = \frac{g_{0i}}{g_{00}}$$

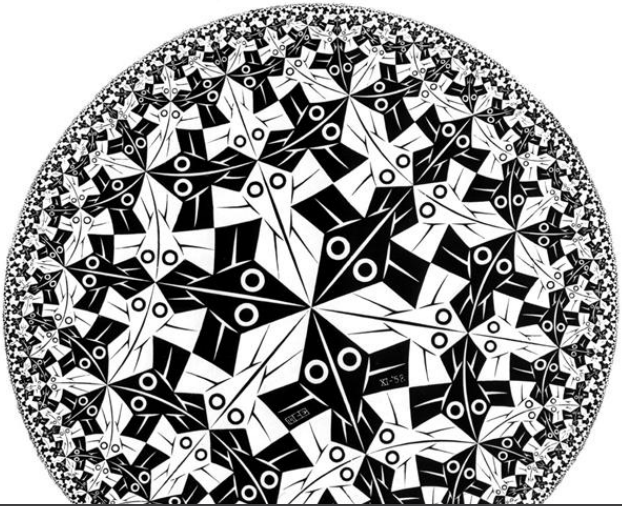
Units: SI; c explicit; ε_0, μ_0 explicit; signature $(-+++)$; $\boldsymbol{g}$ is the magneto-electric vector.



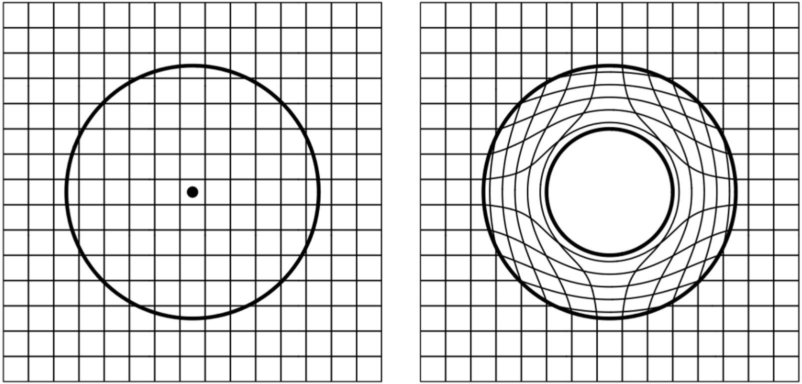


Maxwell is blind to overall scale

$$g_{\mu\nu} \mapsto \Omega^2 g_{\mu\nu}$$



Virtual vs. Physical Space



A coordinate deformation (left) is realised by a graded $\varepsilon_{ij}, \mu_{ij}$ profile (right). Light rays follow geodesics of the *virtual* metric while propagating through an ordinary laboratory sample.

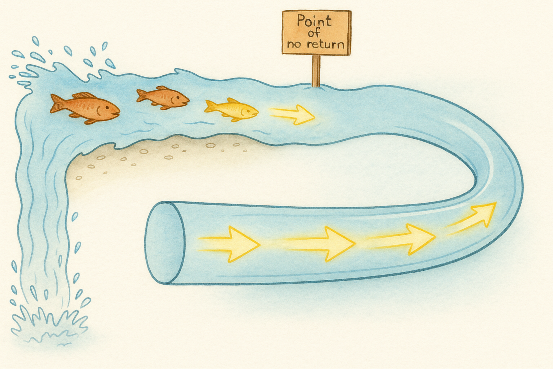
Spacetime transformations in moving media

$$g_i = \frac{g_{0i}}{g_{00}}$$

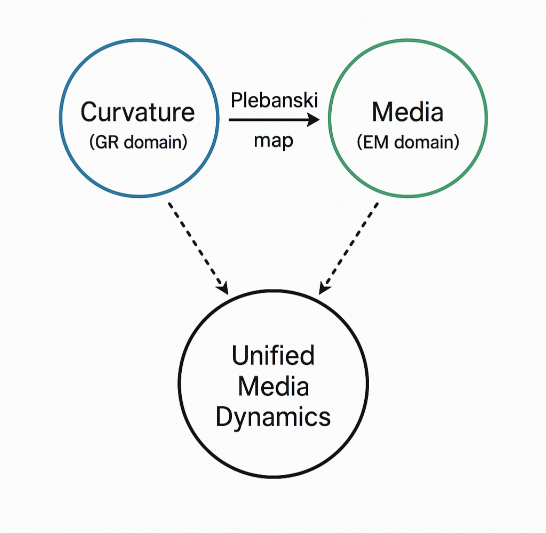
$$g_i^{\mathrm{lab}} = \frac{n^2 - 1}{1 - u^2 n^2 / c^2} u_i$$

Motion mixes **E** and **B** via g_{0i}. Flow/rotation/sound

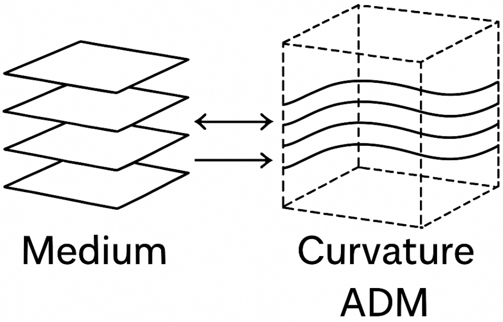
$\Rightarrow$ frame-dragging analogue.



The Big Picture



The Idea



Optical Metric in ADM Form

Transformation optics

$$ds^2 = g_{00}\,dt^2 + 2g_{0i}\,dt\,dx^i + g_{ij}\,dx^i dx^j$$

ADM $(3+1)$ split

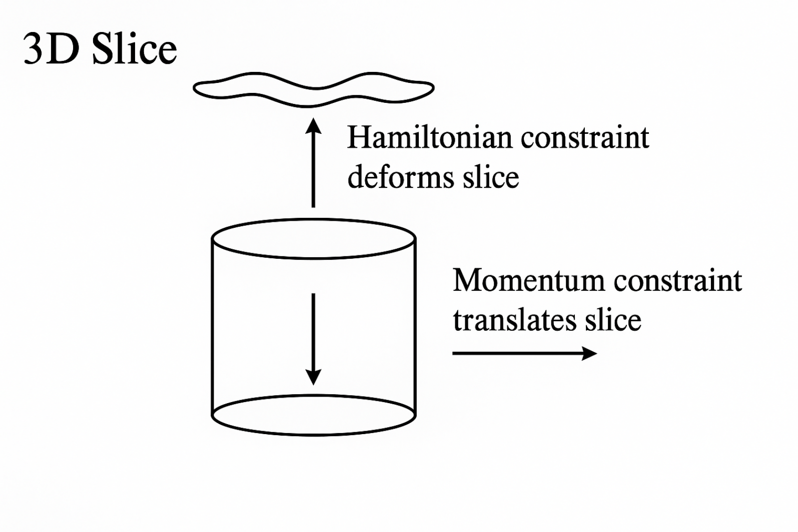
$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

ADM Constraint Equations

$$^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi G\,\rho$$

$$D_j\left(K^{ij} - \gamma^{ij}K\right) = 8\pi G\,j^{\,i}$$

Visualising the ADM Constraints



ADM Evolution Equations

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i$$

$$\partial_t K_{ij} = -D_i D_j \alpha + \alpha \left({}^{(3)}R_{ij} + K K_{ij} - 2 K_{ik} K^k{}_j \right) + \ldots$$

Conformal gauge $g_{00} = -1$, shift $\beta^i = 0$: $\gamma_{ij} = (\det \varepsilon)^{-1/5} \varepsilon_{ij}$.

Self-Gravity of Light

EM source terms

$$\rho = \frac{1}{2c^2}(\boldsymbol{D}\cdot\boldsymbol{E} + \boldsymbol{B}\cdot\boldsymbol{H})$$

$$\boldsymbol{j} = \frac{1}{c^2}\boldsymbol{E}\times\boldsymbol{H}$$

E.g., radiation dominated era.

What about sources?

3+1 Electromagnetism

Thorne & Macdonald (1982)

Membrane formalism

Weak-Field Approximation

Small perturbations

$$g_{ij} = \delta_{ij} + h_{ij}, \qquad |h_{ij}| \ll 1$$

$$\beta^i = 0 \;\Rightarrow\; K_{ij} = -\tfrac{1}{2}\,\partial_t h_{ij}$$

Linearised constraints give wave equations.

Optical Analogue of Gravitational Waves

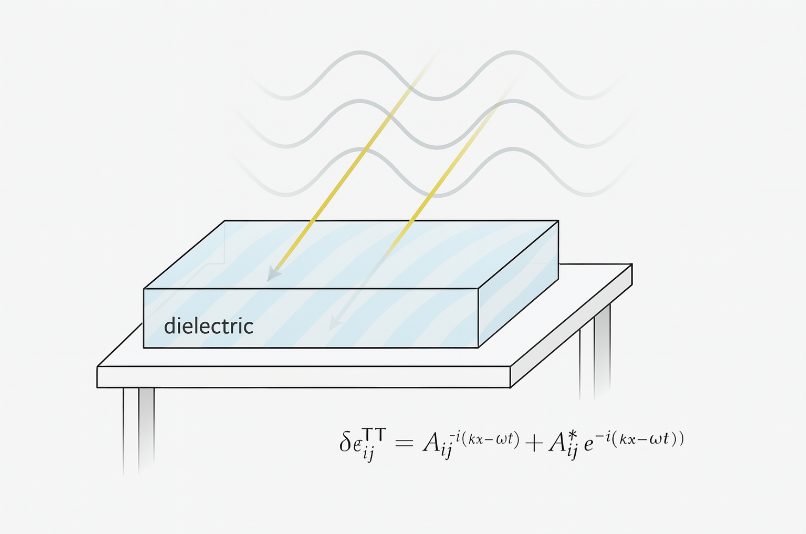
Linearised constraints

$$\partial_j \partial_k h^{jk} - \nabla^2 h = 0$$
$$\partial_j \dot{h}^{ij} - \partial^i \dot{h} = 0$$

TT gauge (two radiative modes)

$$\frac{1}{c^2} \partial_t^2 h_{ij}^{\mathrm{TT}} - \nabla^2 h_{ij}^{\mathrm{TT}} = 0$$

Plane-Wave Solution



Another Einstein↔Maxwell Lens: DGREM (Most et al. 2022)

Maxwell

$$A_\mu \xrightarrow{d} F_{\mu\nu} = dA$$

$$d \star F = \star J$$

homogeneous / inhomogeneous

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Another Einstein↔Maxwell Lens: DGREM (Most et al. 2022)

Maxwell $\Longleftrightarrow$ **DGREM**

$$A_\mu \xrightarrow{d} F_{\mu\nu} = dA$$

$$d \star F = \star J$$

homogeneous / inhomogeneous

$$\theta^{\hat{a}} \xrightarrow{d} F^{\hat{a}} = d\theta^{\hat{a}}$$

$$d\, u^{\hat{a}} = t^{\hat{a}} + \kappa T^{\hat{a}}$$

Nester–Witten / Sparling

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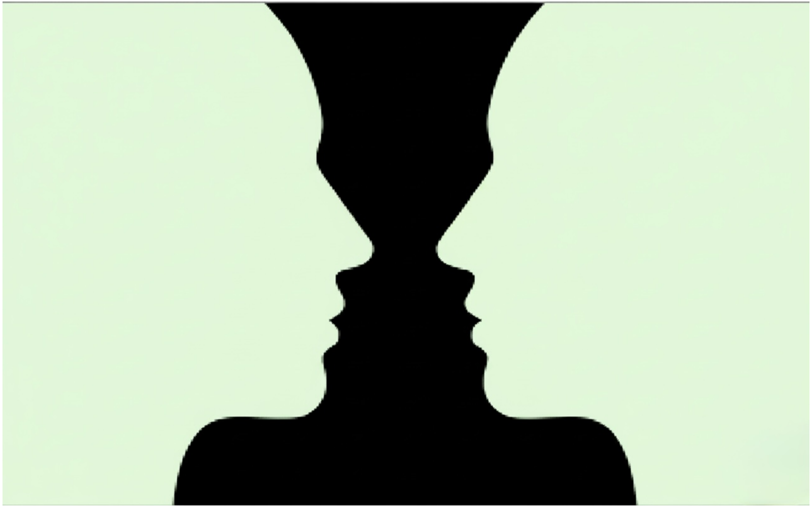
Limitations



Take-Home Message

$$\textbf{Gravity} \iff \textbf{Optics}$$

Take-Home Message

Gravity $\Longleftrightarrow$ Optics

Transformation optics + ADM split

$$g_{\mu\nu} \quad \Longleftrightarrow \quad \left(\varepsilon_{ij},\, \mu_{ij},\, \boldsymbol{g}\right)$$

Take-Home Message

$$\textbf{Gravity} \iff \textbf{Optics}$$

Transformation optics + ADM split

$$g_{\mu\nu} \iff \left(\varepsilon_{ij},\ \mu_{ij},\ \boldsymbol{g}\right)$$

Weak-field limit $\Rightarrow$ tabletop gravitational waves.

Design materials $\equiv$ *design geometry.*

Acknowledgement

This work was carried out under the mentorship and companionship of

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