

Einstein's Equations in Electromagnetic Media

Encoding General Relativity Inside Optical Materials

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ISTA Hosten Group

Requiem for Gravity

From Newton to Einstein

1687: The First Theory of Gravity



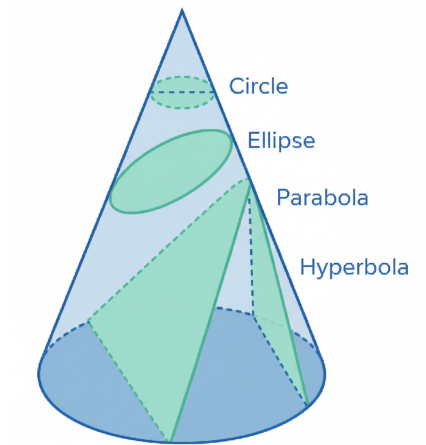
1687: The First Theory of Gravity

$$F = -\frac{GMm}{r^2}$$

The Orbits of the Planets

$$F = -\frac{GMm}{r^2}$$

⇒ conic-section trajectories



Orbits correspond to *conic sections*.

Speed of Light

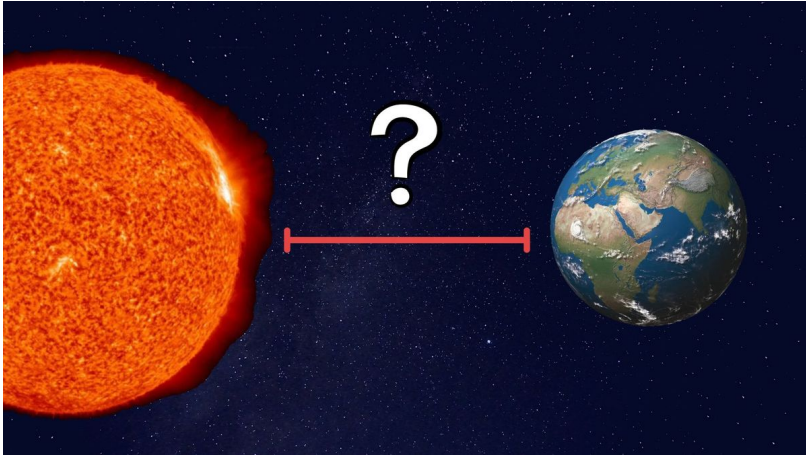
$$c = 299\,792\,458 \text{ m s}^{-1}$$

The Addition of Velocities

$$v_{\text{new}} = \frac{v + u}{1 + \frac{uv}{c^2}}$$

v = speed of object · u = your speed

Einstein's Question



How fast does gravity move?

Einstein's Field Equations

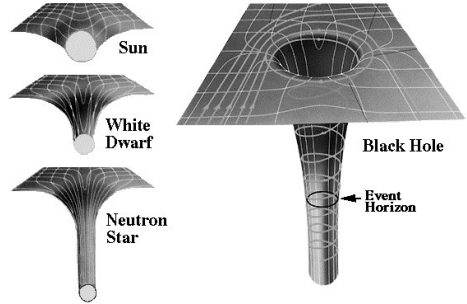
$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

The Mathematics of Curved Space

$$ds^2 = -f(r) c^2 dt^2 + f(r)^{-1} dr^2 + r^2 d\Omega^2$$

$$f(r) = 1 - \frac{2GM}{rc^2}, \quad d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2.$$

Schwarzschild geometry.



Invisibility Devices

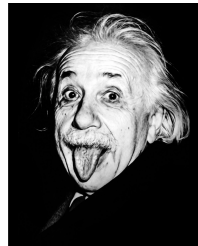
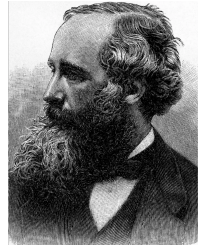


$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \partial_t \mathbf{D} + \mathbf{J}, \quad \nabla \cdot \mathbf{D} = \rho$$

$$\mathbf{D} = \varepsilon_0 \boldsymbol{\varepsilon} \mathbf{E} + \frac{\mathbf{g}}{c} \times \mathbf{H}, \quad \mathbf{B} = \mu_0 \boldsymbol{\mu} \mathbf{H} - \frac{\mathbf{g}}{c} \times \mathbf{E}$$

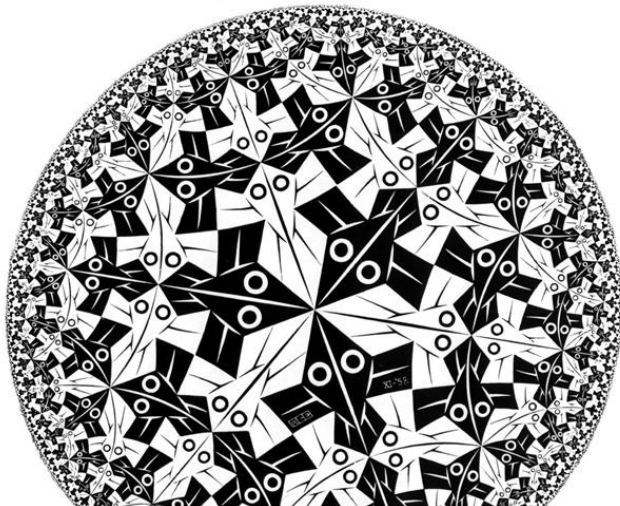
$$\varepsilon^{ij} = \mu^{ij} = -\frac{\sqrt{-g}}{g_{00}} g^{ij}, \quad g_i = \frac{g_{0i}}{g_{00}}$$

Units: SI; c explicit; ε_0, μ_0 explicit; signature $(-+++)$; $\mathbf{g}$ is the magneto-electric vector.

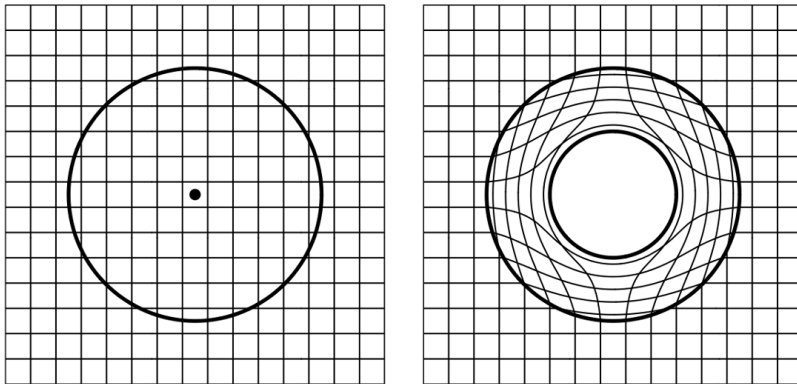


Maxwell is blind to overall scale

$$g_{\mu\nu} \mapsto \Omega^2 g_{\mu\nu}$$



Virtual vs. Physical Space



A coordinate deformation (left) is realised by a graded $\varepsilon_{ij}, \mu_{ij}$ profile (right). Light rays follow geodesics of the *virtual* metric while propagating through an ordinary laboratory sample.

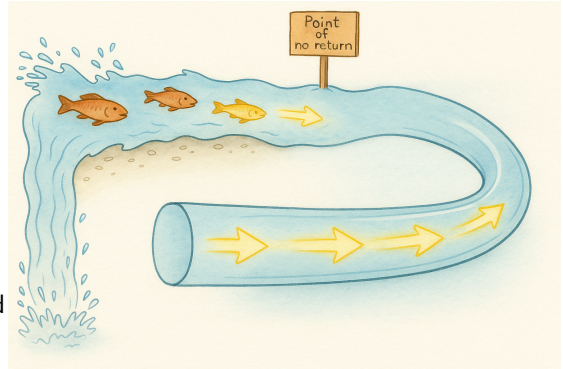
Spacetime transformations in moving media

$$g_i = \frac{g_{0i}}{g_{00}}$$

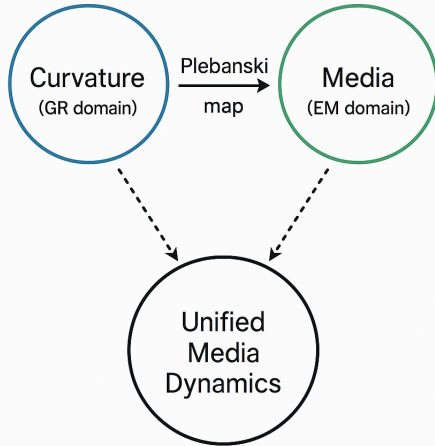
$$g_i^{\text{lab}} = \frac{n^2 - 1}{1 - u^2 n^2 / c^2} u_i$$

Motion mixes **E** and **B** via g_{0i} . Flow/rotation/sound

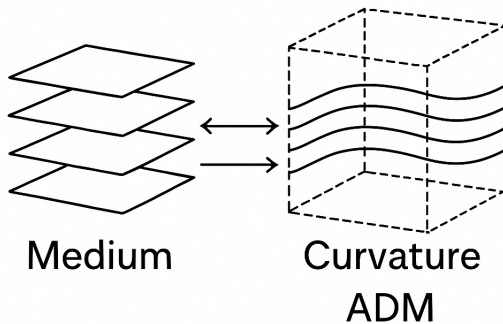
⇒ frame-dragging analogue.



The Big Picture



The Idea



Optical Metric in ADM Form

Transformation optics

$$ds^2 = g_{00} dt^2 + 2g_{0i} dt dx^i + g_{ij} dx^i dx^j$$

ADM (3 + 1) split

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij} (dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

ADM Constraint Equations

$$^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi G \rho$$

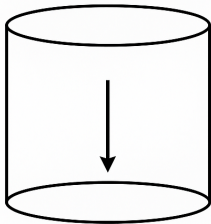
$$D_j(K^{ij} - \gamma^{ij}K) = 8\pi G j^i$$

Visualising the ADM Constraints

3D Slice



Hamiltonian constraint
deforms slice



Momentum constraint
translates slice



ADM Evolution Equations

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i$$

$$\partial_t K_{ij} = -D_i D_j \alpha + \alpha \left({}^{(3)}R_{ij} + K K_{ij} - 2K_{ik} K^k_j \right) + \dots$$

Conformal gauge $g_{00} = -1$, shift $\beta^i = 0$: $\gamma_{ij} = (\det \epsilon)^{-1/5} \epsilon_{ij}$.

Self-Gravity of Light

EM source terms

$$\rho = \frac{1}{2c^2}(D \cdot E + B \cdot H)$$

$$\mathbf{j} = \frac{1}{c^2} \mathbf{E} \times \mathbf{H}$$

E.g., radiation dominated era.

What about sources?

$3+1$ Electromagnetism

Thorne & Macdonald (1982)

Membrane formalism



Weak-Field Approximation

Small perturbations

$$g_{ij} = \delta_{ij} + h_{ij}, \quad |h_{ij}| \ll 1$$

$$\beta^i = 0 \Rightarrow K_{ij} = -\frac{1}{2} \partial_t h_{ij}$$

Linearised constraints give wave equations.

Optical Analogue of Gravitational Waves

Linearised constraints

$$\partial_j \partial_k h^{jk} - \nabla^2 h = 0$$

$$\partial_j \dot{h}^{ij} - \partial^i \dot{h} = 0$$

TT gauge (two radiative modes)

$$\frac{1}{c^2} \partial_t^2 h_{ij}^{\text{TT}} - \nabla^2 h_{ij}^{\text{TT}} = 0$$

Optical Analogue of Gravitational Waves

Hamiltonian constraint

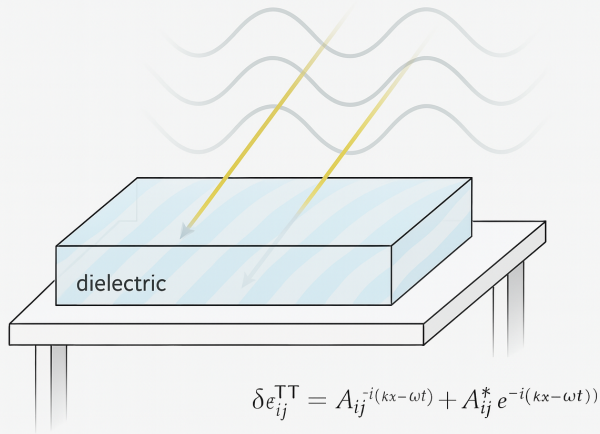
$$\partial_j \partial_k h^{jk} - \nabla^2 h = 0, \quad \text{Momentum : } \partial_j \dot{h}^{ij} - \partial^i \dot{h} = 0.$$

TT-gauge isolates two radiative degrees:

$$\frac{1}{c^2} \partial_t^2 h_{ij}^{\text{TT}} - \nabla^2 h_{ij}^{\text{TT}} = 0.$$

Identify $h_{ij}^{\text{TT}} = \delta \varepsilon_{ij}^{\text{TT}}$ to obtain the optical wave equation for dielectric perturbations.

Plane-Wave Solution



Another Einstein ↔ Maxwell Lens: DGREM (Most et al. 2022)

Maxwell



DGREM

$$A_\mu \xrightarrow{d} F_{\mu\nu} = dA$$

$$d \star F = \star J$$

homogeneous / inhomogeneous

$$\theta^{\hat{a}} \xrightarrow{d} F^{\hat{a}} = d\theta^{\hat{a}}$$

$$d u^{\hat{a}} = t^{\hat{a}} + \kappa T^{\hat{a}}$$

Nester–Witten / Sparling

after Phys. Rev. D 105 (2022) 124038

Gravity $\longleftrightarrow$ Optics

Transformation optics + ADM split

$$g_{\mu\nu} \longleftrightarrow (\varepsilon_{ij}, \mu_{ij}, \mathbf{g})$$

Weak-field limit $\Rightarrow$ tabletop gravitational waves.

Design materials $\equiv$ design geometry.

Limitations



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