1. Spacetime Foliation and Diffeomorphism Invariance

The idea of describing space-time as a spatial state in a given solution space and evolving it with time, reminiscent of the Hamiltonian formulation of quantum mechanics, exists; rightly so, it is called the Hamiltonian formulation of General Relativity, or ADM, after its founders, Arnowitt, Deser, and Misner. However, unlike evolving in Hilbert space, which has a trivial global definition for time, this is not as trivial in general relativity. In Newtonian gravity, this is called Newton–Cartan (NC) spacetime, honoring Henri Cartan for his contributions to our understanding of Newtonian spacetime ($\mathcal{N}$); it embodies this split absolutely. It is structured as a fiber bundle with a base space E^1 (absolute time) and fibers E^3 (Euclidean space). The foliation structure is fixed, which, like in QM, makes the spacetime absolute.

However, in GR the spacetime manifold $(\mathcal{M}, g_{\mu\nu})$ lacks a preferred time structure, meaning the Cauchy Problem, the fancy name for the initial value problem for PDEs, lacks a global definition; hence, not every spacetime can be assumed to be represented this way. But, to a certain degree, every spacetime can, and this formalism of general relativity forms the basis of numerical relativity. Thus, the 3+1 decomposition 'artificially' foliates $\mathcal{M}$ by choosing a scalar time function t:

$$\mathcal{M} \cong \mathbb{R} \times \Sigma_t,$$

defining a stack of 3D spatial hypersurfaces Σ_t equipped with an induced Riemannian metric $\gamma_{ij}(t, x)$.

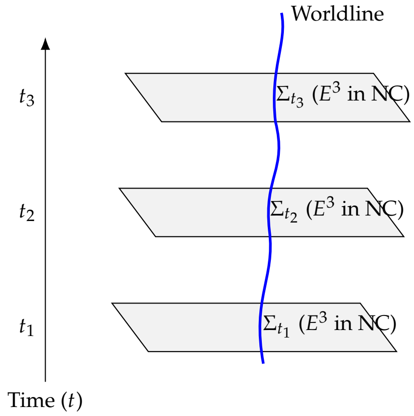

Figure 1: Foliation of spacetime $\mathcal{M}$ into spatial hypersurfaces Σ_t.

2. The Role of Diffeomorphisms.

A key feature of GR is diffeomorphism invariance: the physical structure of spacetime (the geometry G) is independent of the coordinates used. This diffeomorphism invariance created a significant conceptual dilemma during the development of general relativity, especially as first raised by Einstein himself and then continued by Hilbert. Their concern was that, because of this invariance, if a spacetime is curved in the presence of a source, it would be topologically and algebraically equivalent to one with no source. So, they believed this created an impossibility to formulate a proper theory. So, mathematically, every solution of general relativity is pointwise equivalent to every other. However, in GR, what is important is not the relation between these absolute entities, which are grid points of spacetime, but rather how these points are connected to each other. These geometric connections within GR, known as the Christoffel coefficients, are the cause of the description of 'force', not the absolute relation between the points, which is a dramatic difference from the classical picture and a structure lacking in quantum field theory as well. This is also why some physicists don't consider GR as a description of a fundamental source, but instead a pseudoforce. To be more exact, the choice of foliation is a gauge choice. Moving or relabeling the spatial slices is merely a coordinate transformation (a diffeomorphism) that preserves the underlying spacetime structure. So, in simple terms, by aligning the surfaces parallel to each other, I can always return to the original Minkowski spacetime. And this operation I did, moving these surfaces right or left, this "map" is called a coordinate transformation.

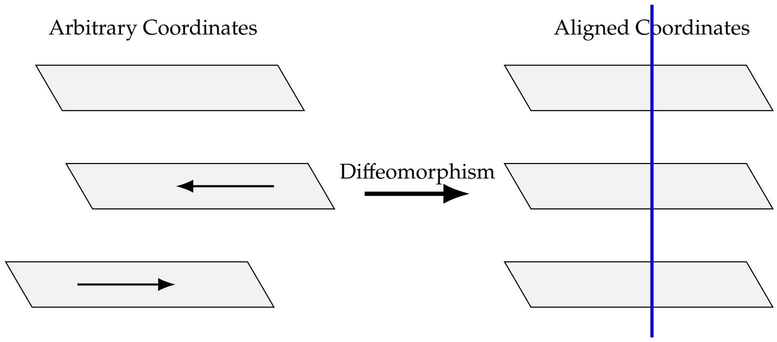

Figure 2: A coordinate transformation (diffeomorphism) can align spatial hypersurfaces. If spacetime is flat, this alignment yields Minkowski coordinates.

3. The 3+1 Decomposition: Kinematics and Metric

The geometry of a chosen foliation is described by the **Lapse function** α and the **Shift vector** β^i.

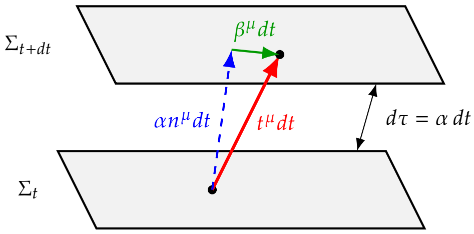

Figure 3: Geometry of the 3+1 decomposition. The coordinate time vector t^μ splits into normal evolution (lapse αn^μ) and tangential evolution (shift β^μ).

The coordinate time vector $t^\mu = (\partial_t)^\mu$ is decomposed as

$$t^\mu = \alpha n^\mu + \beta^\mu,$$

with n^μ the future-pointing unit normal to Σ_t. The lapse relates coordinate time dt to the proper time $d\tau$ measured by an observer moving orthogonally to the slices:

$$\alpha = \left(-g^{\mu\nu}\nabla_\mu t\, \nabla_\nu t\right)^{-1/2}.$$

The shift β^i describes how spatial coordinates are dragged when moving from slice to slice.

The full spacetime metric $g_{\mu\nu}$ is reconstructed from the spatial metric γ_{ij}, the lapse, and the shift:

$$ds^2 = -(\alpha^2 - \beta_i\beta^i)dt^2 + 2\beta_i\, dx^i dt + \gamma_{ij}\, dx^i dx^j, \qquad \beta_i = \gamma_{ij}\beta^j.$$

4. Rigid - Galilean and Newtonian Limits

In GR, the collection of fibers through base space can be viewed as the collection of inertial frames, where a suitable choice of coordinates (an orientation of foliation) can always locally flatten each fibre into an inertial one. The key difference between the trivial bundles of GR and those of Newton–Cartan theory is an extra degree of freedom that allows the spatial foliation to twist and curl dynamically. Hence, Newtonian foliations are absolute entities; lacking the radiative Transverse–Traceless (TT) part of the curvature. Relatedly, both Newtonian and Einsteinian bundles differ from the Galilean bundle (which is rigid) in that the extra freedom now resides in the fibers themselves (allowing for curvature).

We can recover classical spacetime structures by taking specific limits of the 3+1 variables.

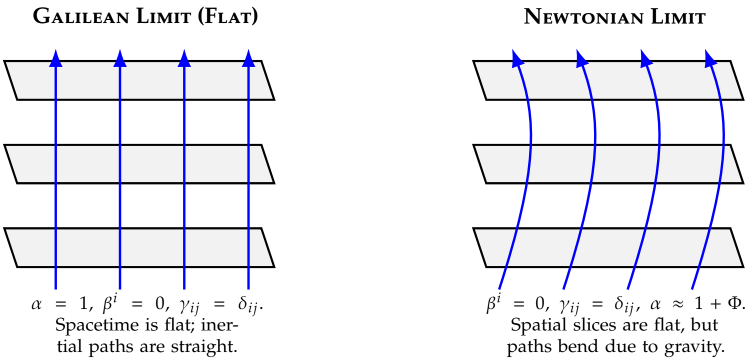

Figure 4: Comparison of the Galilean and Newtonian limits.

In GR, the spatial slices Σ_t can possess intrinsic curvature (non-flat γ_{ij}), and the foliation can twist (non-zero shift β^i). This extra shift is why gravitational waves can be present in Einsteinian gravity but not in Newtonian gravity.